Evaluating convolution sums of the divisor function by quasimodular forms

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EVALUATING CONVOLUTION SUMS OF THE DIVISOR FUNCTION BY QUASIMODULAR FORMS

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A . We provide a systematic method to compute arithmetic sums including some previously computed by Alaca, Alaca, Besge, Cheng, Glaisher, Huard, Lahiri, Lemire, Melfi, Ou, Ramanujan, Spearman and Williams. Our method is based on quasimodular forms. This extension of modular forms has been constructed by Kaneko & Zagier.

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1. I

1.1. **Results.** Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For n and j in $\mathbb{N}^*$ we set

$$\sigma_j(n) := \sum_{d|n} d^j$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_j(n) = 0$. Following [Wil05], for $N \in \mathbb{N}^*$ we define

$$W_N(n) := \sum_{m<n/N} \sigma_1(m)\sigma_1(n - Nm)$$

where m runs through the positive integers $< n/N$. We call W_N the convolution of level N (of the divisor function). We present a method (introduced in [LR05]) to compute these sums using quasimodular forms. The comparison between the results we obtain and the ones already obtained may lead to interesting modular identites (see, for example, remark 1). We insist on the fact that the only goal of this paper is to present a method and we recapitulate, in table 1.1 some of the known results. We hope however that some of our results are new (see, for example, theorems 2, 3, 4 and proposition 9). Whereas the evaluations of $W_N(n)$ for $N \in \{1, 2, 3, 4\}$ given in [HOSW02] are elementary and the ones of $W_N(n)$ for $N \in \{5, \ldots, 9\}$ are analytic in nature and use ideas of Ramanujan, our evaluations are on algebraic nature.

For $N \in \{5, \ldots, 10\}$, we denote by $\Delta_{4,N}$ the unique cuspidal form spanning the cuspidal subspace of the modular forms of weight 4 on $\Gamma_0(N)$ with Fourier expansion[1] $\Delta_{4,N}(z) = e^{2\pi iz} + O\left(e^{4\pi iz}\right)$. We define

$$\Delta_{4,N}(z) =: \sum_{n=1}^{+\infty} \tau_{4,N}(n)e^{2\pi inz}.$$

We also write

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{+\infty} \left[1 - e^{2\pi inz}\right]^{24} =: \sum_{n=1}^{+\infty} \tau(n)e^{2\pi inz}$$

for the unique primitive form of weight 12 on $\mathrm{SL}(2, \mathbb{Z})$.

[1]In this paper, "Fourier expansion" always means "Fourier expansion at the cusp ∞".

Level N	Who	Where
1	Besge (Liouville), Glaisher, Ramanujan	[Bes62], [Gla85], [Ram16]
2, 3, 4	Huard, Ou, Spearman & Williams	[HOSW02]
5, 7	Lemire & Williams	[LW05]
6	Alaca & Williams	[AW]
8	Williams	[Wil]
9	Williams	[Wil05]
12	Alaca, Alaca & Williams	[AAWc]
16	Alaca, Alaca & Williams	[AAWa]
18	Alaca, Alaca & Williams	[AAWd]
24	Alaca, Alaca & Williams	[AAWb]

T 1. Some previous computations of W_N

Theorem 1. *Let $n \in \mathbb{N}^*$, then*

$$W_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n),$$

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right),$$

$$W_3(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{12}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{3}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{3}\right),$$

$$\begin{aligned} W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) \\ + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right), \end{aligned}$$

$$\begin{aligned} W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) - \frac{1}{20}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{5}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{5}\right) \\ - \frac{1}{130}\tau_{4,5}(n), \end{aligned}$$

$$\begin{aligned} W_6(n) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{24}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{6}\right) \\ + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{6}\right) - \frac{1}{120}\tau_{4,6}(n), \end{aligned}$$

$$\begin{aligned} W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{28}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{7}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{7}\right) \\ - \frac{1}{70}\tau_{4,7}(n), \end{aligned}$$

$$\begin{aligned} W_8(n) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) - \frac{1}{32}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{8}\right) \\ + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{8}\right) - \frac{1}{64}\tau_{4,8}(n), \end{aligned}$$

$$W_9(n) = \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{36}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{9}\right) - \frac{1}{54}\tau_{4,9}(n).$$

$$W_{10}(n) = \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{10}\right) - \frac{1}{40}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{10}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{10}\right) - \frac{1}{120}\tau_{4,10}(n) - \frac{3}{260}\tau_{4,5}(n) - \frac{3}{65}\tau_{4,5}\left(\frac{n}{2}\right).$$

For dimensional reasons, the forms $\Delta_{4,N}$ are primitive forms for $N \in \{5, \ldots, 10\}$, meaning that they are eigenvalues of all the Hecke operators and that their Fourier expansion begins with $e^{2\pi iz} + O(e^{4\pi iz})$. It follows that the arithmetic functions $\tau_{4,N}$ are multiplicative and satisfy the relation (2) (see below). Following [Koi84], one obtains

$$\begin{aligned}
\Delta_{4,5}(z) &= [\Delta(z)\Delta(5z)]^{1/6}\,,\\
\Delta_{4,6}(z) &= [\Delta(z)\Delta(2z)\Delta(3z)\Delta(6z)]^{1/12}\,,\\
\Delta_{4,8}(z) &= [\Delta(2z)\Delta(4z)]^{1/6}\,,\\
\Delta_{4,9}(z) &= [\Delta(3z)]^{1/3}\,,
\end{aligned}$$

whereas $\Delta_{4,7}$ and $\Delta_{4,10}$ *are not* products of the η function.

However, using M [BCP97] (see [Ste04] for the algorithms based on the computation of the spectrum of Hecke operators on modular symbols), one can compute their Fourier coefficients (see tables 2 and 3).

Remark 1. The independant computation of W_7 by Lemire & Williams [LW05] implies that

$$\Delta_{4,7}(z) = \left[\left(\Delta(z)^2\Delta(7z)\right)^{1/3} + 13\left(\Delta(z)\Delta(7z)\right)^{1/2} + 49\left(\Delta(z)\Delta(7z)^2\right)^{1/3}\right]^{1/3}.$$

This provide an alternative method to compute the function $\tau_{4,7}$. It is likely that, following [LW05] to evaluate W_{10} we could get an expression of $\Delta_{4,10}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,7}(n)$	1	−1	−2	−7	16	2	−7	15	−23	−16	−8
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,7}(n)$	14	28	7	−32	41	54	23	−110	−112	14	8

T 2. First Fourier coefficients of $\Delta_{4,7}$

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,10}(n)$	1	2	−8	4	5	−16	−4	8	37	10	12
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,10}(n)$	−32	−58	−8	−40	16	66	74	−100	20	32	24

T 3. First Fourier coefficients of $\Delta_{4,10}$

In each of our previous examples, we did not leave the field of rational numbers. This might not happen, since the primitive forms do not necessarily have rational coefficients. However, every evaluation will make use of totally real algebraic numbers for coefficients since the extension of $\mathbb{Q}$ by the Fourier coefficients of a primitive form is finite and totally real [Shi72, Proposition 1.3]. To illustrate that fact, we shall evaluate the convolution sum of level 11 and 13. The set of primitive modular forms of weight 4 on $\Gamma_0(11)$ has two elements. The coefficients of these two primitive forms are in $\mathbb{Q}(t)$ where t is a root of X^2-2X-2 (see § 2.8 for the use of a method founded in [Zag92]). Each primitive form is determined by the beginning of its Fourier expansion:

$$\Delta_{4,11,1}(z) = e^{2\pi i z} + (2-t)e^{4\pi i z} + O(e^{6\pi i z})$$
$$\Delta_{4,11,2}(z) = e^{2\pi i z} + te^{4\pi i z} + O(e^{6\pi i z}).$$

We denote by $\tau_{4,11,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,11,i}$. The two primitive forms, and hence their Fourier coefficients, are conjugate by $t \mapsto 2-t$ (see, for example, [DI95] for the general result and § 2.8 for the special case needed here). We give in § 2.8 a way to compute the functions $\tau_{4,11,i}$ for $i \in \{1,2\}$.

n	1	2	3	4	5
$\tau_{4,11,1}(n)$	1	$-t+2$	$4t-5$	$-2t-2$	$-8t+9$
n	6	7	8	9	10
$\tau_{4,11,1}(n)$	$5t-18$	$4t+6$	$10t-16$	$-8t+30$	$-9t+34$
n	11	12	13	14	15
$\tau_{4,11,1}(n)$	-11	$-14t-6$	$20t+20$	$-6t+4$	$12t-109$

T 4. First Fourier coefficients of $\Delta_{4,11,1}$ where $t^2-2t-2=0$

Theorem 2. *Let $n \in \mathbb{N}^*$. Then*

$$W_{11}(n) = \frac{5}{1464}\sigma_3(n) + \frac{605}{1464}\sigma_3\left(\frac{n}{11}\right) - \frac{1}{44}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{11}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{11}\right) - \frac{2t+43}{4026}\tau_{4,11,1}(n) + \frac{2t-47}{4026}\tau_{4,11,2}(n).$$

Remark 2. We have

$$-\frac{2t+43}{4026}\tau_{4,11,1}(n) + \frac{2t-47}{4026}\tau_{4,11,2}(n) = \mathrm{tr}_{\mathbb{Q}(t)/\mathbb{Q}}\left[-\frac{2t+43}{4026}\tau_{4,11,1}(n)\right] \in \mathbb{Q}.$$

The set of primitive modular forms of weight 4 on $\Gamma_0(13)$ has three elements. One of them, we note $\Delta_{4,13,1}$, has Fourier coefficients in $\mathbb{Q}$. The two others, we note $\Delta_{4,13,2}$ and $\Delta_{4,13,3}$, have Fourier coefficients in $\mathbb{Q}(u)$ where u is a root of X^2-X-4. Each of these two primitive form is determined by the beginning of its Fourier expansion:

$$\Delta_{4,13,2}(z) = e^{2\pi i z} + (1-u)e^{4\pi i z} + O(e^{6\pi i z})$$
$$\Delta_{4,13,3}(z) = e^{2\pi i z} + ue^{4\pi i z} + O(e^{6\pi i z}).$$

We denote by $\tau_{4,13,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,13,i}$. The two primitive forms $\Delta_{4,13,2}$ and $\Delta_{4,13,3}$, and hence their Fourier coefficients, are conjugate by $u \mapsto 1-u$ (see, for example, [DI95]). We compute table 5 by use of M .

n	1	2	3	4	5
$\tau_{4,13,2}(n)$	1	$-u+1$	$3u+1$	$-u-3$	$-u-1$
n	6	7	8	9	10
$\tau_{4,13,2}(n)$	$-u-11$	$-11u+1$	$11u-7$	$15u+10$	$u+3$
n	11	12	13	14	15
$\tau_{4,13,2}(n)$	$-12u+46$	$-13u-15$	-13	$-u+45$	$-7u-13$

T 5. First Fourier coefficients of $\Delta_{4,13,2}$ where $u^2-u-4=0$

Theorem 3. *Let $n \in \mathbb{N}^*$. Then*

$$W_{13}(n) = \frac{1}{408}\sigma_3(n) + \frac{169}{408}\sigma_3\left(\frac{n}{13}\right) - \frac{1}{52}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{13}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{13}\right) + \frac{u-6}{442}\tau_{4,13,2}(n) - \frac{u+5}{442}\tau_{4,13,3}(n).$$

Remark 3. We have

$$\frac{u-6}{442}\tau_{4,13,2}(n) - \frac{u+5}{442}\tau_{4,13,3}(n) = \mathrm{tr}_{\mathbb{Q}(u)/\mathbb{Q}}\left[\frac{u-6}{442}\tau_{4,13,2}(n)\right] \in \mathbb{Q}.$$

The set of primitive modular forms of weight 4 on $\Gamma_0(14)$ has two elements. Both have coefficients in $\mathbb{Q}$ and we can distinguish them by the beginning of their Fourier expansion:

$$\Delta_{4,14,1}(z) = e^{2\pi iz} + 2e^{4\pi iz} + O(e^{6\pi iz})$$
$$\Delta_{4,14,2}(z) = e^{2\pi iz} - 2e^{4\pi iz} + O(e^{6\pi iz}).$$

We denote by $\tau_{4,14,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,14,i}$ and give in §2.10 a method to compute these coefficients and get tables 6 and 7.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,14,1}(n)$	1	2	-2	4	-12	-4	7	8	-23	-24	48
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,14,1}(n)$	-8	56	14	24	16	-114	-46	2	-48	-14	96

T 6. First Fourier coefficients of $\Delta_{4,14,1}$

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,14,2}(n)$	1	-2	8	4	-14	-16	-7	-8	37	28	-28
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,14,2}(n)$	32	18	14	-112	16	74	-74	80	-56	-56	56

T 7. First Fourier coefficients of $\Delta_{4,14,2}$

Theorem 4. *Let $n \in \mathbb{N}^*$. Then*

$$W_{14}(n) = \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right) - \frac{1}{56}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{14}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{14}\right) - \frac{3}{350}\tau_{4,7}(n) - \frac{6}{175}\tau_{4,7}\left(\frac{n}{2}\right) - \frac{1}{84}\tau_{4,14,1}(n) - \frac{1}{200}\tau_{4,14,2}(n).$$

Remark 4. The fact that for each $N \in \{12, 16, 18, 20, 24\}$ there exists only one primitive form of weight 4 over $\Gamma_0(N)$ and at most one of weight 2 implies that the only modular forms appearing in the evaluation of the corresponding W_N have rational coefficients.

Our method, with the introduction of Dirichlet characters, also allows to recover another result of Williams [Wil05, Theorem 1.2] which extended a result of Melfi [Mel98, Theorem 2, (7)]. This result is theorem 5. For $b \in \mathbb{N}^*$ and $a \in \{0, \dots, b-1\}$, we define

$$S[a,b](n) := \sum_{\substack{m=0 \\ m \equiv a \pmod{b}}}^{n} \sigma_1(m)\sigma_1(n-m).$$

We compute $S[i,3]$ for $i \in \{0,1,2\}$. Our result uses the primitive Dirichlet character χ_3 defined by

$$\chi_3(n) := \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ -1 & \text{if } n \equiv -1 \pmod{3} \end{cases}$$

for all $n \in \mathbb{N}^*$.

Theorem 5. *Let $n \in \mathbb{N}^*$, then*

$$S[0,3](n) = \frac{11}{72}\sigma_3(n) + \frac{25}{18}\sigma_3\left(\frac{n}{3}\right) - \frac{9}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{4}n\sigma_1(n) - n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{4}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\left[1 + \delta(3 \mid n)\right]\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n),$$

$$\begin{aligned} S[1,3](n) = {} & \frac{19}{144}\sigma_3(n) + \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right) \\ & - \frac{1}{8}n\sigma_1(n) - \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) \\ & + \frac{1}{24}\delta(3 \mid n-1)\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n) \end{aligned}$$

and

$$\begin{aligned} S[2,3](n) = {} & \frac{19}{144}\sigma_3(n) - \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right) \\ & - \frac{1}{8}n\sigma_1(n) + \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) \\ & + \frac{1}{24}\delta(3 \mid n-2)\sigma_1(n) - \frac{1}{9}\tau_{4,9}(n). \end{aligned}$$

where $\delta(3 \mid n)$ is 1 *if* 3 *divides n and* 0 *otherwise.*

We next consider convolutions of different divisor sums and recover results of Melfi [Mel98, Theorem 2, (9), (10)] completed by Huard, Ou, Spearman & Williams [HOSW02, Theorem 6] and Cheng & Williams [CW05]. We shall use the unique cuspidal form $\Delta_{8,2}$ spanning the cuspidal subspace of the modular forms of weight 8 on $\Gamma_0(2)$ with Fourier expansion $\Delta_{8,2}(z) = e^{2\pi i z} + O\left(e^{4\pi i z}\right)$. Using [Koi84], we have

$$\Delta_{8,2}(z) = [\eta(z)\eta(2z)]^8.$$

We define

$$\Delta_{8,2}(z) =: \sum_{n=1}^{+\infty} \tau_{8,2}(n) e^{2\pi i n z}.$$

This is again a primitive form, hence the arithmetic function $\tau_{8,2}$ is multiplicative and satisfies the relation (2) (see below).

Theorem 6. *Let $n \in \mathbb{N}^*$. Then*

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_3(n-k) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_3(k) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_3\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_3(n-2k) = \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{16}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1\left(\frac{n}{2}\right).$$

Moreover,

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_5(n-k) = \frac{5}{126}\sigma_7(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_5(n-2k) =$$
$$\frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{24}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}\tau_{8,2}(n),$$

and

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_5(k) =$$
$$\frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{12}n\sigma_5\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}\tau_{8,2}(n).$$

In theorem 6, the first and fourth identites are due to Ramanujan [Ram16]. The second and third ones are due to Huard, Ou, Spearman & Williams [HOSW02, Theorem 6]. The fifth and sixth ones are due to Cheng & Williams [CW05]. Some other identities of the same type may be found in [CW05] and [Ram16].

Our method also allows to evaluate sums of Lahiri type

$$(1) \quad S[(a_1,\dots,a_r),(b_1,\dots,b_r),(N_1,\dots,N_r)](n) :=$$
$$\sum_{\substack{(m_1,\dots,m_r)\in\mathbb{N}^r \\ m_1+\dots+m_r=n}} m_1^{a_1}\dots m_r^{a_r}\sigma_{b_1}\left(\frac{m_1}{N_1}\right)\dots\sigma_{b_r}\left(\frac{m_r}{N_r}\right)$$

[Lah46], [Lah47], [HOSW02, §3] where the a_i are nonnegative integers, the N_i are positive integers and the b_i are odd positive integers. To simplify the notations, we introduce

$$S[(a_1,\dots,a_r),(b_1,\dots,b_r)](n) := S[(a_1,\dots,a_r),(b_1,\dots,b_r),(1,\dots,1)](n).$$

For example, we prove the following.

Theorem 7. *Let $n \in \mathbb{N}^*$. Then*

$$S[(0,1,1),(1,1,1)](n) = \frac{1}{288}n^2\sigma_5(n) - \frac{1}{72}n^3\sigma_3(n) + \frac{1}{288}n^2\sigma_3(n) + \frac{1}{96}n^4\sigma_1(n) - \frac{1}{288}n^3\sigma_1(n).$$

and if

$$\begin{aligned}
A(n) &= -\frac{48}{5}n^2\sigma_9(n)\\
B(n) &= 128n^3\sigma_7(n)\\
C(n) &= -80n^2\sigma_7(n) - 600n^4\sigma_5(n)\\
D(n) &= 648n^3\sigma_5(n) + \frac{8208}{7}n^5\sigma_3(n)\\
E(n) &= -144n^2\sigma_5(n) - \frac{11232}{7}n^4\sigma_3(n) - \frac{3456}{5}n^6\sigma_1(n)\\
F(n) &= 576n^3\sigma_3(n) + \frac{5184}{5}n^5\sigma_1(n)\\
G(n) &= -432n^4\sigma_1(n) - 48n^2\sigma_3(n)\\
H(n) &= 48n^3\sigma_1(n)\\
I(n) &= \frac{8}{35}n\tau(n)\\
J(n) &= -\frac{8}{35}\tau(n)
\end{aligned}$$

then

$$\begin{aligned}
&-24^5S[(0,0,0,1,1),(1,1,1,1,1)](n) =\\
&\qquad A(n)+B(n)+C(n)+D(n)+E(n)+F(n)+G(n)+H(n)+I(n)+J(n).
\end{aligned}$$

The first identity of theorem 7 is due to Lahiri [Lah46, (5.9)] and an elementary proof had been given by Huard, Ou, Spearman & Williams [HOSW02]. The second identity is due to Lahiri [Lah47].

We continue our evaluations by the more complicated sum $S[(0,1),(1,1),(2,5)]$. The reason why it is more difficult is that the underlying space of new cuspidal modular forms has dimension 3.

The space of newforms of weight 6 on $\Gamma_0(10)$ has dimension 3. Let $\{\Delta_{6,10,i}\}_{1\leq i\leq 3}$ be the unique basis of primitive forms with

$$\begin{aligned}
\Delta_{6,10,1}(z) &= e^{2\pi iz} + 4e^{4\pi iz} + 6e^{6\pi iz} + O(e^{8\pi iz}),\\
\Delta_{6,10,2}(z) &= e^{2\pi iz} - 4e^{4\pi iz} + 24e^{6\pi iz} + O(e^{8\pi iz}),\\
\Delta_{6,10,3}(z) &= e^{2\pi iz} - 4e^{4\pi iz} - 26e^{6\pi iz} + O(e^{8\pi iz}).
\end{aligned}$$

Again, by [Koi84], we know that these functions *are not* products of the η function. We denote by $\tau_{6,10,i}(n)$ the nth Fourier coefficient of $\Delta_{6,10,i}$. The functions $\tau_{6,10,i}$ are multiplicative and we show in § 5.2 how to establish tables 8, 9 and 10.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,1}(n)$	1	4	6	16	−25	24	−118	64	−207	−100	192
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,1}(n)$	96	1106	−472	−150	256	762	−828	−2740	−400	−708	768

T 8. First Fourier coefficients of $\Delta_{6,10,1}$

We also need the unique primitive form

$$\Delta_{6,5}(z) =: \sum_{n=1}^{+\infty}\tau_{6,5}(n)e^{2\pi inz}$$

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,2}(n)$	1	−4	24	16	25	−96	−172	−64	333	−100	132
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,2}(n)$	384	−946	688	600	256	−222	−1332	500	400	−4128	−528

T 9. First Fourier coefficients of $\Delta_{6,10,2}$

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,3}(n)$	1	−4	−26	16	−25	104	−22	−64	433	100	−768
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,3}(n)$	−416	−46	88	650	256	378	−1732	1100	−400	572	3072

T 10. First Fourier coefficients of $\Delta_{6,10,3}$

of weight 6 on $\Gamma_0(5)$. It is not a product of the η function, and we show in § 5.2 how to establish table 11.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,5}(n)$	1	2	−4	−28	25	−8	192	−120	−227	50	−148
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,5}(n)$	112	286	384	−100	656	−1678	−454	1060	−700	−768	−296

T 11. First Fourier coefficients of $\Delta_{6,5}$

Proposition 8. *Let $n \in \mathbb{N}^*$. Define*

$$\begin{aligned}
A(n) &= \frac{12}{13}n\sigma_3(n) + \frac{48}{13}n\sigma_3\left(\frac{n}{2}\right) + \frac{300}{13}n\sigma_3\left(\frac{n}{5}\right) + \frac{1200}{13}n\sigma_3\left(\frac{n}{10}\right)\\
B(n) &= -\frac{48}{5}n^2\sigma_1\left(\frac{n}{2}\right) - 48n^2\sigma_1\left(\frac{n}{5}\right)\\
C(n) &= 24n\sigma_1\left(\frac{n}{5}\right)\\
D(n) &= \frac{12}{5}n\tau_{4,10}(n) - \frac{216}{65}n\tau_{4,5}(n) - \frac{864}{65}n\tau_{4,5}\left(\frac{n}{2}\right)\\
E(n) &= \frac{108}{35}\tau_{6,5}(n) + \frac{864}{35}\tau_{6,5}\left(\frac{n}{2}\right) - \frac{24}{5}\tau_{6,10,1}(n) + \frac{12}{7}\tau_{6,10,2}(n).
\end{aligned}$$

Then

$$5 \times 24^2 \sum_{\substack{(a,b)\in\mathbb{N}^{*2}\\ 2a+5b=n}} b\sigma_1(a)\sigma_1(b) = A(n) + B(n) + C(n) + D(n) + E(n).$$

We shall now evaluate $S[(1,1),(1,1),(1,5)]$ since it constitutes an exemple leaving the rational field. Let v be one of the two roots of $X^2 - 20X + 24$. There exist three primitive forms of weight 8 on $\Gamma_0(5)$ determined by the beginning of their Fourier expansion:

$$\begin{aligned}
\Delta_{8,5,1}(z) &= e^{2\pi iz} - 14e^{4\pi iz} + O(e^{6\pi iz})\\
\Delta_{8,5,2}(z) &= e^{2\pi iz} + (20 - v)e^{4\pi iz} + O(e^{6\pi iz})\\
\Delta_{8,5,3}(z) &= e^{2\pi iz} + ve^{4\pi iz} + O(e^{6\pi iz}).
\end{aligned}$$

The function $\Delta_{8,5,3}$ is obtained from $\Delta_{8,5,2}$ by the conjugation $(v \mapsto 20 - v)$ of $\mathbb{Q}(v)$ on the Fourier coefficients. We denote by $\tau_{8,5,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{8,5,i}$ and show in § 5.3 how to compute them.

n	1	2	3	4	5	6	7	8	9
$\tau_{8,5,1}(n)$	1	-14	-48	68	125	672	-1644	840	117
n	10	11	12	13	14	15	16	17	18
$\tau_{8,5,1}(n)$	-1750	172	-3264	3862	23016	-6000	-20464	-12254	-1638

T 12. First Fourier coefficients of $\Delta_{8,5,1}$

n	1	2	3	4	5
$\tau_{8,5,2}(n)$	1	$-v+20$	$8v-70$	$-20v+248$	-125
n	6	7	8	9	10
$\tau_{8,5,2}(n)$	$70v-1208$	$-56v+510$	$-120v+1920$	$160v+1177$	$125v-2500$
n	11	12	13	14	15
$\tau_{8,5,2}(n)$	$-400v+6272$	$184v-13520$	$608v-4310$	$-510v+8856$	$-1000v+8750$

T 13. First Fourier coefficients of $\Delta_{8,5,2}$ where $v^2-20v+24=0$

Proposition 9. *Let $n \in \mathbb{N}^*$. Define*

$$\begin{aligned}
A(n) &= \frac{24}{13}n^2\sigma_3(n) + \frac{600}{13}n^2\sigma_3\left(\frac{n}{5}\right)\\
B(n) &= -\frac{24}{5}n^3\sigma_1(n) - 24n^3\sigma_1\left(\frac{n}{5}\right)\\
C(n) &= -\frac{288}{325}n^2\tau_{4,5}(n)\\
D(n) &= \frac{792+12v}{475}\tau_{8,5,2}(n) + \frac{1032-12v}{475}\tau_{8,5,3}(n).
\end{aligned}$$

Then

$$5\times 24^2 \sum_{\substack{(a,b)\in\mathbb{N}^{*2}\\ a+5b=n}} ab\sigma_1(a)\sigma_1(b) = A(n)+B(n)+C(n)+D(n).$$

Remark 5. The two terms in the right hand side of the definition of $D(n)$ in proposition 9 being conjugate, we have

$$D(n) = \operatorname{tr}_{\mathbb{Q}(v)/\mathbb{Q}}\left[\frac{792+12v}{475}\tau_{8,5,2}(n)\right] \in \mathbb{Q}.$$

To stay in the field of rational numbers, we could have used the fundamental fact that, for every even $k>0$ and every integer $N\geq 1$, the space of cuspidal forms of weight k on $\Gamma_0(N)$ has a basis whose elements have a Fourier expansion with integer coefficients [Shi94, Theorem 3.52]. However, the coefficients of these Fourier expansions are often not multiplicative: this is a good reason to leave $\mathbb{Q}$.

Remark 6. If τ_* is one of our τ functions, its values are the Fourier coefficients of a primitive form (of weight k on $\Gamma_0(N)$ say). It therefore satisfies the following multiplicativity relation

$$\tau_*(mn) = \sum_{\substack{d\mid(m,n)\\ (d,N)=1}} \mu(d)d^{k-1}\tau_*\left(\frac{m}{d}\right)\tau_*\left(\frac{n}{d}\right). \tag{2}$$

Remark 7. When we found some, we give some expression to compute the various Fourier coefficients that we need. This is however somewhat *ad hoc* and never needed since we only need to compute a finite number of coefficients and can then use the algorithms provided by algorithmic number theory [BCP97], [Ste04].

Thanks – While a preliminary version of this paper was in circulation, K.S. Williams kindly informed me of the papers [AW], [LW05], [Wil], [AAWc], [AAWa], [AAWd], [AAWb] and [CW05]. I respectfully thank him for having made these papers available to me. The final version of this paper was written during my stay at the Centre de Recherches Mathématiques de Montréal which provided me with very good working conditions. I thank Andrew Granville and Chantal David for their invitation.

1.2. **Method.** Since our method is based on quasimodular forms (anticipated by Rankin [Ran56] and formally introduced by Kaneko & Zagier in [KZ95]), we briefly recall the basics on these functions, referring to [MR05] and [LR05] for the details.

Define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{Z}^4,\ ad - bc = 1,\ N \mid c \right\}$$

for all integers $N \geq 1$. In particular, $\Gamma_0(1)$ is $\mathrm{SL}(2, \mathbb{Z})$. Denote by $\mathscr{H}$ the Poincaré upper half plane:

$$\mathscr{H} = \{z \in \mathbb{C} \colon\ \Im m\, z > 0\}.$$

Definition 10. *Let $N \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $s \in \mathbb{N}^*$. A holomorphic function*

$$f \colon \mathscr{H} \to \mathbb{C}$$

is a quasimodular form of weight k, depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathscr{H}$ such that

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz + d}\right)^i \tag{3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

Here is what is meant by the requirement for f_s to be holomorphic at the cusps. One can show [MR05, Lemme 119] that if f satisfies the quasimodularity condition (3), then f_s satisfies the modularity condition

$$(cz + d)^{-(k-2s)} f_s\left(\frac{az + b}{cz + d}\right) = f_s(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Asking f_s to be holomorphic at the cusps is asking that, for all $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(1)$, the function

$$z \mapsto (\gamma z + \delta)^{-(k-2s)} f_s\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

has a Fourier expansion of the form

$$\sum_{n=0}^{+\infty} \widehat{f_{s,M}}(n) e\left(\frac{nz}{u_M}\right)$$

where

$$u_M = \inf\{u \in \mathbb{N}^* \colon T^u \in M^{-1}\Gamma_0(N)M\}.$$

In other words, f_s is automatically a modular function and is required to be more than that, a modular form of weight $k - 2s$ on $\Gamma_0(N)$. It follows that if f is a quasimodular form of weight k and depth s, non identically vanishing, then k is even and $s \leq k/2$.

A fundamental quasimodular form is the Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24\sum_{n=1}^{+\infty}\sigma_1(n)e^{2\pi inz}.$$

It is a quasimodular form of weight 2, depth 1 on $\Gamma_0(1)$ (see, for example, [Ser77, Chapter 7]).

We shall denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N)]$ the space of quasimodular forms of weight k, depth $\leq s$ on $\Gamma_0(N)$ and $M_k[\Gamma_0(N)] = \widetilde{M}_k^{\leq 0}[\Gamma_0(N)]$ the space of modular forms of weight k on $\Gamma_0(N)$. The space $\widetilde{M}_k^{\leq k/2}[\Gamma_0(N)]$ is graded by the depth.

Our method for theorem 1 is to remark that the function

$$\begin{aligned} H_N(z) &= E_2(z)E_2(Nz) \\ &= 1 - 24\sum_{n=1}^{+\infty}\left[\sigma_1(n) + \sigma_1\left(\frac{n}{N}\right)\right]e^{2\pi inz} + 576\sum_{n=1}^{+\infty}W_N(n)e^{2\pi inz} \end{aligned}$$

is a quasimodular form of weight 4, depth 2 on $\Gamma_0(N)$ that we linearise using the following lemma.

Lemma 11. *Let $k \geq 2$ even. Then*

$$\widetilde{M}_k^{\leq k/2}[\Gamma_0(N)] = \bigoplus_{i=0}^{k/2-1} D^iM_{k-2i}[\Gamma_0(N)] \oplus \mathbb{C}D^{k/2-1}E_2.$$

We have set

$$D \coloneqq \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}.$$

Let $\{B_k\}_{k\in\mathbb{N}}$ be the sequence of rational numbers defined by its exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{+\infty}B_k\frac{t^k}{k!}.$$

We shall use the Eisenstein series to express the basis we need:

$$E_{k,N}(z) \coloneqq 1 - \frac{2k}{B_k}\sum_{n=1}^{+\infty}\sigma_{k-1}(n)e^{2\pi inNz} \in M_k[\Gamma_0(N)]$$

for all $k \in 2\mathbb{N}^* + 2$, $N \in \mathbb{N}^*$. If $N = 1$ we simplify by writing $E_k \coloneqq E_{k,N}$. For weight 2 forms, we shall need

$$\Phi_{a,b}(z) = \frac{1}{b-a}\left[bE_2(bz) - aE_2(az)\right] \in M_2[\Gamma_0(b)]$$

for all $b > 1$ and $a \mid b$.

Let χ be a Dirichlet character. If f satisfies all of what is needed to be a quasimodular form except (3) being replaced by

$$(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right) = \chi(d)\sum_{i=0}^{n}f_i(z)\left(\frac{c}{cz+d}\right)^i,$$

then one says that f is a quasimodular form of weight k, depth s and character χ on $\Gamma_0(N)$. (In particular, we require f_s to be a modular form of character χ). We denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi]$ the vector space of quasimodular forms of weight k, depth $\leq s$ and character χ on $\Gamma_0(N)$. If $\chi = \chi_0$ is a principal character of modulus dividing N, then $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi] = \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$.

If $f \in \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$, then f has a Fourier expansion with coefficients $\{\widehat{f}(n)\}_{n\in\mathbb{N}}$. We define the twist of f by the Dirichlet character χ as

$$f \otimes \chi(z) = \sum_{n=0}^{+\infty} \chi(n)\widehat{f}(n)e^{2\pi inz}.$$

In [LR05, Proposition 9], we proved the following proposition.

Proposition 12. *Let χ be a primitive Dirichlet character of conductor m. Let f be a quasimodular form of weight k and depth s on $\Gamma_0(N)$. Then $f \otimes \chi$ is a quasimodular form of weight k, depth less than or equal to s and character χ^2 on $\Gamma_0\left(\operatorname{lcm}(N, m^2)\right)$.*

Remark 8. The condition of primitivity of the character may be replaced by the condition of non vanishing of its Gauss sum.

The proof of theorem 5 follows from the linearisation of $E_2 \cdot E_2 \otimes \chi_3$.

Theorems 6 and 7 follow from the linearisation of derivatives of forms of type $E_j E_{k,N}$.

1.3. **Generalisation of the results.** For $N \geq 1$ and $k \geq 2$, let $A_{N,k}^*$ be the set of triples (ψ, ϕ, t) such that ψ is a primitive Dirichlet character of modulus L, ϕ is a primitive Dirichlet character of modulus M and t is an integer such that $tLM \mid N$ (and $tLM \neq 1$ if $k = 2$) with the extra condition

$$\psi\phi(n) = \begin{cases} 1 & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N}^*). \tag{4}$$

We write $\mathbf{1}$ for the primitive character of modulus 1 (the constant function $n \mapsto 1$). We extend the definition of σ_k: for k and n in $\mathbb{N}^*$ we set

$$\sigma_k^{\psi,\phi}(n) := \sum_{d\mid n} \psi\left(\frac{n}{d}\right)\phi(d)d^k$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_k^{\psi,\phi}(n) = 0$. If M is the modulus of the primitive character ϕ, we define the sequence $\{B_k^\phi\}_{k\in\mathbb{N}}$ by its exponential generating function

$$\sum_{c=0}^{M-1} \phi(c)\frac{te^{ct}}{e^{Mt} - 1} = \sum_{k=0}^{+\infty} B_k^\phi \frac{t^k}{k!}.$$

For any $(\psi, \phi, t) \in A_{N,k}^*$, define

$$E_k^{\psi,\phi}(z) := \delta(\psi = \mathbf{1}) - \frac{2k}{B_k^\phi}\sum_{n=1}^{+\infty} \sigma_{k-1}^{\psi,\phi}(n)e^{2\pi inz}$$

and

$$E_{k,t}^{\psi,\phi}(z) := \begin{cases} E_k^{\psi,\phi}(tz) & \text{if } (k, \psi, \phi) \neq (2, \mathbf{1}, \mathbf{1}) \\ E_2^{\mathbf{1},\mathbf{1}}(z) - tE_2^{\mathbf{1},\mathbf{1}}(tz) & \text{otherwise} \end{cases}$$

where $\delta(\psi = \mathbf{1})$ is 1 if $\psi = \mathbf{1}$ and 0 otherwise.

For $N \geq 1$ and $k \geq 2$ even, the set

$$\left\{E_{k,t}^{\psi,\phi} : (\psi, \phi, t) \in A_{N,k}^*\right\}$$

is a basis for the orthogonal subspace (called Eisenstein subspace, the scalar product being the Petersson one) of the cuspidal subspace $S_k[\Gamma_0(N)]$ of $M_k[\Gamma_0(N)]$ [DS05, Chapter 4].

Moreover, by Atkin-Lehner-Li theory [DS05, Chapter 5], a basis for $S_k[\Gamma_0(N)]$ is

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}} \alpha_d\left(H_k^*[\Gamma_0(M)]\right)$$

where α_d is

$$\begin{array}{cccc}\alpha_d & : & M_k[\Gamma_0(M)] & \to & M_k[\Gamma_0(M)]\\ & & f & \mapsto & [z\mapsto f(dz)]\end{array}$$

and $H_k^*[\Gamma_0(M)]$ is the set of primitive forms of weight k on $\Gamma_0(M)$.

A corollary is the following generalisation of theorems 1 and 6. If f is a modular form, we denote by $\{\widehat{f}(n)\}_{n\in\mathbb{N}}$ the sequence of its Fourier coefficients.

Proposition 13. *Let $N \geq 1$. There exist scalars $a_{\psi,\phi,t}$, $a_{M,d,f}$ and a such that, for all $n \geq 1$, we have*

$$\begin{aligned}W_N(n) = &\sum_{(\psi,\phi,t)\in A^*_{N,4}} a_{\psi,\phi,t}\sigma_3^{\psi,\phi}\left(\frac{n}{t}\right) + \sum_{(\psi,\phi,t)\in A^*_{N,2}} a_{\psi,\phi,t}n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right) + an\sigma_1(n)\\ &+ \sum_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}}\sum_{f\in H_4^*[\Gamma_0(M)]} a_{M,d,f}\widehat{f}\left(\frac{n}{d}\right) + \sum_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}}\sum_{f\in H_2^*[\Gamma_0(M)]} a_{M,d,f}n\widehat{f}\left(\frac{n}{d}\right)\\ &+\frac{1}{24}\sigma_1(n)+\frac{1}{24}\sigma_1\left(\frac{n}{N}\right).\end{aligned}$$

More generally, for any $N \geq 1$ and any even $\ell \geq 4$, the arithmetic functions

$$n \mapsto \sum_{k<n/N}\sigma_1(n-kN)\sigma_{\ell-1}(k) - \frac{B_\ell}{2\ell}\sigma_1(n) - \frac{1}{24}\sigma_{\ell-1}\left(\frac{n}{N}\right)$$

and

$$n \mapsto \sum_{k<n/N}\sigma_1(k)\sigma_{\ell-1}(n-kN) - \frac{B_\ell}{2\ell}\sigma_1\left(\frac{n}{N}\right) - \frac{1}{24}\sigma_{\ell-1}(n)$$

are linear combinations of the sets of functions

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell+2}}\left\{n\mapsto\sigma_{\ell+1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell}}\left\{n\mapsto n\sigma_{\ell-1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}}\bigcup_{f\in H^*_{\ell+2}[\Gamma_0(M)]}\left\{n\mapsto\widehat{f}\left(\frac{n}{d}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^*\\ dM|N}}\bigcup_{f\in H^*_{\ell}[\Gamma_0(M)]}\left\{n\mapsto n\widehat{f}\left(\frac{n}{d}\right)\right\}.$$

The same allows to generalise theorem 5. If $b \geq 1$ is an integer, denote by $X(b)$ the set of Dirichlet characters of modulus b. By orthogonality, we have

$$S[a,b](n) = \frac{1}{\varphi(b)}\sum_{\chi\in X(b)}\overline{\chi(a)}\sum_{m=1}^{n-1}\chi(m)\sigma_1(m)\sigma_1(n-m).$$

It follows that the function to be considered is now

$$\frac{1}{\varphi(b)}\sum_{\chi\in X(b)}\overline{\chi(a)}E_2\cdot E_2\otimes\chi.$$

We restrict to b squarefree so that the Gauss sum associates to any character of modulus b is non vanishing. For $N \geq 1$, let $\chi_N^{(0)}$ be the principal character of modulus N. For $\chi \in X(b)$, we define $A^*_{N,k,\chi}$ as $A^*_{N,k}$ except we replace condition (4) by

$$\psi\phi = \chi_N^{(0)}\chi.$$

Then, similarly to the proposition 13, we have the following proposition.

Proposition 14. *Let $b \geq 1$ squarefree and $a \in [0, b-1]$ be integers. Then the function*

$$n \mapsto S[a,b](n) - \frac{1}{24}[\delta(b \mid a) + \delta(b \mid n-a)]\sigma_1(n)$$

is a linear combination of the set of functions

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t)\in A^*_{N,4,\chi}} \left\{n \mapsto \sigma_3^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t)\in A^*_{N,2,\chi}} \left\{n \mapsto n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_4^*[\Gamma_0(M),\chi_N^{(0)}\chi]} \left\{n \mapsto \widehat{f}\left(\frac{n}{d}\right)\right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_2^*[\Gamma_0(M),\chi_N^{(0)}\chi]} \left\{n \mapsto n\widehat{f}\left(\frac{n}{d}\right)\right\}$$

$$\{n \mapsto n\sigma_1(n)\}$$

where N is the least common multiple of 2 *and* b^2 *and* $\delta(b \mid n-a)$ *is* 1 *if* $n \equiv a \pmod b$ *and* 0 *otherwise.*

2. C

2.1. Level 3. By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(3)] = M_4[\Gamma_0(3)] \oplus DM_2[\Gamma_0(3)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(3)]$ has dimension 2 and is spanned by the two linearly independent forms E_4 and $E_{4,3}$. The vector space $M_2[\Gamma_0(3)]$ has dimension 1 and is spanned by $\Phi_{1,3}$. Computing the first Fourier coefficients, we therefore find that

$$H_3 = \frac{1}{10}E_4 + \frac{9}{10}E_{4,3} + 4D\Phi_{1,3} + 4DE_2. \tag{5}$$

Comparing with the Fourier expansion in (5) leads to the corresponding result in theorem 1.

2.2. Level 5. By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(5)] = M_4[\Gamma_0(5)] \oplus DM_2[\Gamma_0(5)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(5)]$ has dimension 3 and is spanned by the linearly independent forms E_4, $E_{4,5}$ and $\Delta_{4,5}$. The vector space $M_2[\Gamma_0(5)]$ has dimension 1 and is spanned by $\Phi_{1,5}$. Computing the first Fourier coefficients, we therefore find that

$$H_5 = \frac{1}{26}E_4 + \frac{25}{26}E_{4,5} - \frac{288}{65}\Delta_{4,5} + \frac{24}{5}D\Phi_{1,5} + \frac{12}{5}DE_2 \tag{6}$$

Comparing with the Fourier expansion in (6) leads to the corresponding result in theorem 1.

2.3. **Level 6.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(6)] = M_4[\Gamma_0(6)] \oplus DM_2[\Gamma_0(6)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(6)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,2}$,$E_{4,3}$ $E_{4,6}$ and $\Delta_{4,6}$. The vector space $M_2[\Gamma_0(6)]$ has dimension 3 and is spanned by the three linearly independent forms $\Phi_{1,2}$, $\Phi_{1,3}$ and $\Phi_{3,6}$. Computing the first Fourier coefficients, we therefore find that

$$H_6 = \frac{1}{50}E_4 + \frac{2}{25}E_{4,2} + \frac{9}{50}E_{4,3} + \frac{18}{25}E_{4,6} - \frac{24}{5}\Delta_{4,6} + 2D\Phi_{1,3} + 3D\Phi_{3,6} + 2DE_2. \tag{7}$$

Comparing with the Fourier expansion in (7) leads to the corresponding result in theorem 1.

2.4. **Level 7.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(7)] = M_4[\Gamma_0(7)] \oplus DM_2[\Gamma_0(7)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(7)]$ has dimension 3 and is spanned by the three linearly independent forms E_4, $E_{4,7}$ and $\Delta_{4,7}$. The vector space $M_2[\Gamma_0(7)]$ has dimension 1 and is spanned by the form $\Phi_{1,7}$. Computing the first Fourier coefficients, we therefore find that

$$H_7 = \frac{1}{50}E_4 + \frac{49}{50}E_{4,7} - \frac{288}{35}\Delta_{4,7} + \frac{36}{7}D\Phi_{1,7} + \frac{12}{7}DE_2. \tag{8}$$

Comparing with the Fourier expansion in (8) leads to the corresponding result in theorem 1.

2.5. **Level 8.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(8)] = M_4[\Gamma_0(8)] \oplus DM_2[\Gamma_0(8)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(8)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,2}$, $E_{4,4}$, $E_{4,8}$ and $\Delta_{4,8}$. The vector space $M_2[\Gamma_0(8)]$ has dimension 3 and is spanned by the forms $\Phi_{1,4}$, $\Phi_{1,8}$ and

$$\Phi_{1,4,2} \coloneqq z \mapsto \Phi_{1,4}(2z).$$

Computing the first Fourier coefficients, we therefore find that

$$H_8 = \frac{1}{80}E_4 + \frac{3}{80}E_{4,2} + \frac{3}{20}E_{4,4} + \frac{4}{5}E_{4,8} - 9\Delta_{4,8} + \frac{21}{4}D\Phi_{1,8} + \frac{3}{2}DE_2. \tag{9}$$

Comparing with the Fourier expansion in (9) leads to the corresponding result in theorem 1.

2.6. **Level 9.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(9)] = M_4[\Gamma_0(9)] \oplus DM_2[\Gamma_0(9)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(9)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_4 \otimes \chi_3$, $E_{4,3}$, $E_{4,9}$ and $\Delta_{4,9}$. The vector space $M_2[\Gamma_0(9)]$ has dimension 3 and is spanned by the forms $\Phi_{1,3}$, $\Phi_{1,3} \otimes \chi_3$ and $\Phi_{1,9}$.

Computing the first Fourier coefficients, we therefore find that

$$H_9 = \frac{1}{90}E_4 + \frac{4}{45}E_{4,3} + \frac{9}{10}E_{4,9} - \frac{32}{3}\Delta_{4,9} + \frac{16}{3}D\Phi_{1,9} + \frac{4}{3}DE_2. \tag{10}$$

Comparing with the Fourier expansion in (10) leads to the corresponding result in theorem 1.

2.7. **Level 10.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(10)] = M_4[\Gamma_0(10)] \oplus DM_2[\Gamma_0(10)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(10)]$ has dimension 7 and is spanned by the seven linearly independent forms E_4, $E_{4,2}$, $E_{4,5}$, $E_{4,10}$, $\Delta_{4,10}$, $\Delta_{4,5}$ and

$$F_{4,5,2} \coloneqq z \mapsto \Delta_{4,5}(2z).$$

The vector space $M_2[\Gamma_0(10)]$ has dimension 3 and is spanned by the forms $\Phi_{1,10}$, $\Phi_{1,5}$ and

$$\Phi_{1,5,2} \coloneqq z \mapsto \Phi_{1,5}(2z).$$

Computing the first Fourier coefficients, we therefore find that

$$H_{10} = \frac{1}{130}E_4 + \frac{2}{65}E_{4,2} + \frac{5}{26}E_{4,5} + \frac{10}{13}E_{4,10} - \frac{24}{5}\Delta_{4,10} - \frac{432}{65}\Delta_{4,5} - \frac{1728}{65}F_{4,5,2} + \frac{27}{5}D\Phi_{1,10} + \frac{6}{5}DE_2. \tag{11}$$

Comparison with the Fourier expansion in (11) leads to the corresponding result in theorem 1.

2.8. **Level 11.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(11)] = M_4[\Gamma_0(11)] \oplus DM_2[\Gamma_0(11)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(11)]$ has dimension 4 and is spanned by the four linearly independent forms E_4, $E_{4,11}$, $\Delta_{4,11,1}$ and $\Delta_{4,11,2}$. Let F_1 be the parabolic form of weight 4 and level 11 given by

$$F_1(z) = [\Delta(z)\Delta(11z)]^{1/6} = e^{4\pi iz} - 4e^{6\pi iz} + 2e^{8\pi iz} + 8e^{10\pi iz} + O(e^{12\pi iz}).$$

Let T_2 be the Hecke operator of level 11 given by

$$T_2 : \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imz} \mapsto \sum_{m\in\mathbb{Z}} \left[\sum_{\substack{d\in\mathbb{N} \\ d|(m,2) \\ (d,11)=1}} d^{k-1}\widehat{f}\left(\frac{2m}{d^2}\right) \right] e^{2\pi imz}.$$

It sends a parabolic form of weight 4 and level 11 to another one. Let

$$F_2 = T_2F_1 = e^{2\pi iz} + 2e^{4\pi iz} - 5e^{6\pi iz} - 2e^{8\pi iz} + 9e^{10\pi iz} + O(e^{12\pi iz}).$$

There exists λ_1 and λ_2 such that

$$\Delta_{4,11,1} = F_2 + \lambda_1F_1 \quad \text{and} \quad \Delta_{4,11,2} = F_2 + \lambda_2F_1.$$

For $j \in \{1,2\}$, it follows that

$$\tau_{4,11,j}(2) = 2 + \lambda_j \text{ and } \tau_{4,11,j}(4) = -2 + 2\lambda_j.$$

Since $\Delta_{4,11,j}$ is primitive, it satisfies (2) hence $\lambda_j^2 - 2\lambda_j - 2 = 0$. In other words

$$X^2 - 2X - 2 = (X - \lambda_1)(X - \lambda_2).$$

This provides a way to compute the Fourier coefficients of $\Delta_{4,11,1}$ and $\Delta_{4,11,2}$ from the ones of Δ and proves that these coefficients live in $\mathbb{Q}(t)$ where t is a root of $X^2 - 2X - 2$.

The vector space $M_2[\Gamma_0(11)]$ has dimension 2 and is spanned by the form $\Phi_{1,11}$ and its unique primitive form

$$\Delta_{2,11} = [\Delta(z)\Delta(11z)]^{1/12}.$$

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{2,11}(n)$	1	−2	−1	2	1	2	−2	0	−2	−2	1
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{2,11}(n)$	−2	4	4	−1	−4	−2	4	0	2	2	−2

T 14. First Fourier coefficients of $\Delta_{2,11}$

Computing the first Fourier coefficients, we therefore find that

$$H_{11} = \frac{1}{122}E_4 + \frac{121}{122}E_{4,11} - \frac{192t+4128}{671}\Delta_{4,11,1} + \frac{192t-4512}{671}\Delta_{4,11,2} + \frac{60}{11}D\Phi_{1,11} + \frac{12}{11}DE_2. \tag{12}$$

Comparison with the Fourier expansion in (12) leads to theorem 2.

2.9. **Level 13.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(13)] = M_4[\Gamma_0(13)] \oplus DM_2[\Gamma_0(13)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(13)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,13}$, $\Delta_{4,13,1}$, $\Delta_{4,13,2}$ and $\Delta_{4,13,3}$. The vector space $M_2[\Gamma_0(13)]$ has dimension 1 and is spanned by the form $\Phi_{1,13}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,13,1}(n)$	1	−5	−7	17	−7	35	−13	−45	22	35	−26
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,13,1}(n)$	−119	13	65	49	89	77	−110	−126	−119	91	130

T 15. First Fourier coefficients of $\Delta_{4,13,1}$

Computing the first Fourier coefficients, we therefore find that

$$H_{13} = \frac{1}{170}E_4 + \frac{169}{170}E_{4,13} + \frac{288u-1728}{221}\Delta_{4,13,2} - \frac{288u+1440}{221}\Delta_{4,13,3} + \frac{72}{13}D\Phi_{1,13} + \frac{12}{13}DE_2. \tag{13}$$

Comparison with the Fourier expansion in (13) leads to the theorem 3.

2.10. **Level 14.** By lemma 11, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(14)] = M_4[\Gamma_0(14)] \oplus DM_2[\Gamma_0(14)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(14)]$ has dimension 8 and is spanned by the eight linearly independent forms E_4, $E_{4,2}$, $E_{4,7}$, $E_{4,14}$, $\Delta_{4,7}$,

$$F_{4,7,2} \colon z \mapsto \Delta_{4,7}(2z),$$

$\Delta_{4,14,1}$ and $\Delta_{4,14,2}$. Another basis of the subspace of parabolic forms is $\Delta_{4,7}$, $F_{4,7,2}$, $\Delta_{2,14}^2$ and $\Delta_{2,14}\Phi_{1,14}$ where

$$\Delta_{2,14}(z) := [\Delta(z)\Delta(2z)\Delta(7z)\Delta(14z)]^{1/24}$$

is the unique primitive form of weight 2 on $\Gamma_0(14)$. We echelonise this second basis by defining

$$\begin{aligned}
J_1 &= -\frac{11}{28}\Delta_{4,7} - \frac{22}{7}F_{4,7,2} + \frac{11}{7}\Delta_{2,14}^2 + \frac{39}{28}\Delta_{2,14}\Phi_{1,14} && = e^{2\pi iz} + O\left(e^{10\pi iz}\right)\\
J_2 &= -\frac{13}{56}\Delta_{4,7} + \frac{1}{7}F_{4,7,2} + \frac{3}{7}\Delta_{2,14}^2 + \frac{13}{56}\Delta_{2,14}\Phi_{1,14} && = e^{4\pi iz} + O\left(e^{10\pi iz}\right)\\
J_3 &= \frac{13}{56}\Delta_{4,7} + \frac{19}{14}F_{4,7,2} - \frac{13}{14}\Delta_{2,14}^2 - \frac{13}{56}\Delta_{2,14}\Phi_{1,14} && = e^{6\pi iz} + O\left(e^{10\pi iz}\right)\\
J_4 &= -\frac{13}{56}\Delta_{4,7} - \frac{6}{7}F_{4,7,2} + \frac{3}{7}\Delta_{2,14}^2 + \frac{13}{56}\Delta_{2,14}\Phi_{1,14} && = e^{8\pi iz} + O\left(e^{10\pi iz}\right).
\end{aligned}$$

We then have

$$\Delta_{4,14,j} = J_1 + b_jJ_2 + c_jJ_3 + d_jJ_4.$$

From $\tau_{4,14,j}(4) = \tau_{4,14,j}(2)^2$ we deduce $d_j = b_j^2$. Then, from $\tau_{4,14,j}(6) = \tau_{4,14,j}(2)\tau_{4,14,j}(3)$ and $\tau_{4,14,j}(8) = \tau_{4,14,j}(2)\tau_{4,14,j}(4)$ we respectively deduce

$$\begin{aligned}
2b_j + b_jc_j + 2c_j &= -4\\
b_j^3 - b_j^2 + 6b_j + 4c_j &= 8
\end{aligned}$$

that is

$$c_j = -\frac{1}{4}b_j^3 + \frac{1}{4}b_j^2 - \frac{3}{2}b_j + 2$$

and

$$(b_j - 2)(b_j + 2)(b_j^2 + b_j + 8) = 0.$$

Since the coefficients of $\Delta_{4,14,j}$ are all totally real, we must have

$$\begin{aligned}
\Delta_{4,14,1} &= J_1 + 2J_2 - 2J_3 + 4J_4\\
\Delta_{4,14,2} &= J_1 - 2J_2 + 8J_3 + 4J_4.
\end{aligned}$$

Finally,

$$\Delta_{4,14,1} = -\frac{9}{4}\Delta_{4,7} - 9F_{4,7,2} + 6\Delta_{2,14}^2 + \frac{13}{4}\Delta_{2,14}\Phi_{1,14} \tag{14}$$

$$\Delta_{4,14,2} = \Delta_{4,7} + 4F_{4,7,2} - 5\Delta_{2,14}^2. \tag{15}$$

Equations (14) and (15) allow to compute the first terms of the sequences $\tau_{4,14,1}$ and $\tau_{4,14,2}$. The vector space $M_2[\Gamma_0(14)]$ has dimension 4 and is spanned by the forms $\Phi_{1,7}$, $\Phi_{1,14}$, $\Phi_{2,14}$ and its unique primitive form $\Delta_{2,14}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{2,14}(n)$	1	−1	−2	1	0	2	1	−1	1	0	0
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{2,14}(n)$	−2	−4	−1	0	1	6	−1	2	0	−2	0

T 16. First Fourier coefficients of $\Delta_{2,14}$

Computing the first Fourier coefficients, we therefore find that

$$\begin{aligned}
H_{14} = \frac{1}{250}E_4 + \frac{2}{125}E_{4,2} + \frac{49}{250}E_{4,7} + \frac{98}{125}E_{4,14} - \frac{864}{175}\Delta_{4,7} - \frac{3456}{175}F_{4,7,2}\\
- \frac{48}{7}\Delta_{4,14,1} - \frac{72}{25}\Delta_{4,14,2} + \frac{39}{7}D\Phi_{1,14} + \frac{6}{7}DE_2.
\end{aligned} \tag{16}$$

Comparison with the Fourier expansion in (16) leads to theorem 4.

2.11. **Convolutions of level 1, 2, 4.** The convolutions of level dividing 4 were evaluated in [LR05, Proposition 7]. We obtained

$$W_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n)$$

from the equality

$$E_2^2 = E_4 + 12DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(1)]$;

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right)$$

from the equality

$$H_2 = \frac{1}{5}E_4 + \frac{4}{5}E_{4,2} + 3D\Phi_{1,2} + 6DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(2)]$; and

$$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right).$$

from the equality

$$H_4 = \frac{1}{20}E_4 + \frac{3}{20}E_{4,2} + \frac{4}{5}E_{4,4} + \frac{9}{2}D\Phi_{1,4} + 3DE_2.$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(4)]$.

3. T

Let $\chi_3^{(0)}$ be the principal character of modulus 3. Remarking that

$$S[0,3](n) = \sum_{a+b=n}\sigma_1(a)\sigma_1(b) - \sum_{a+b=n}\chi_3^{(0)}(a)\sigma_1(a)\sigma_1(b),$$

we consider $E_2^2 - E_2(E_2\otimes\chi_3^{(0)})$. Since $E_2\otimes\chi_3^{(0)} \in \widetilde{M}_2^{\leq 1}[\Gamma_0(9),\chi_3^{(0)}] = \widetilde{M}_2^{\leq 1}[\Gamma_0(9)]$, we have

$$E_2^2 - E_2(E_2\otimes\chi_3^{(0)}) \in \widetilde{M}_4^{\leq 2}[\Gamma_0(9)].$$

We use the same method and notations as in §2.6. We compute

$$E_2^2 - E_2(E_2\otimes\chi_3^{(0)}) = \frac{11}{30}E_4 + \frac{10}{3}E_{4,3} - \frac{27}{10}E_{4,9} + 32\Delta_{4,9} + 16D\Phi_{1,3} - 16D\Phi_{1,9} + 12DE_2.$$

The evaluation of $S[1,3]$ given in theorem 5 follows by comparison of the Fourier expansions.

We compute $S[1,3]$ after having remarked that

$$\frac{\chi_3^{(0)}(n)+\chi_3(n)}{2} = \begin{cases}1 & \text{if } n\equiv 1 \pmod 3\\ 0 & \text{otherwise.}\end{cases}$$

Hence, the function to be linearised here is

$$\frac{1}{2}E_2[E_2\otimes\chi_3^{(0)} + E_2\otimes\chi_3]$$

whose nth Fourier coefficient ($n\in\mathbb{N}^*$) is

$$-24\delta(3\mid n-1)\sigma_1(n) + 576S[1,3](n).$$

This is again a quasimodular form in $\widetilde{M}_4^{\leq 2}[\Gamma_0(9)]$, and as in §2.6, we linearise it as

$$\frac{19}{60}E_4 + \frac{1}{20}E_4\otimes\chi_3 - \frac{5}{3}E_{4,3} + \frac{27}{20}E_{4,9} + 32\Delta_{4,9} - 8D\Phi_{1,3} - 6D(\Phi_{1,3}\otimes\chi_3) + 8D\Phi_{1,9}.$$

The evaluation of $S[1,3]$ given in theorem 5 follows by comparison of the Fourier expansions.

The evaluation of $S[2,3]$ follows immediately from

$$S[0,3](n) + S[1,3](n) + S[2,3](n) = W_1(n)$$

and theorem 1.

4. O M

The first three identities of theorem 6 are a direct consequence of the following ones:

$$E_2E_4 \in \widetilde{M}_6^{\leq 1}[\Gamma_0(1)] = M_6[\Gamma_0(1)] \oplus DM_4[\Gamma_0(1)] = \mathbb{C}E_6 \oplus \mathbb{C}DE_4,$$

and

$$\begin{aligned} E_2E_{4,2},\ E_4E_{2,2} \in \widetilde{M}_6^{\leq 1}[\Gamma_0(2)] &= M_6[\Gamma_0(2)] \oplus DM_4[\Gamma_0(2)] \\ &= \mathbb{C}E_6 \oplus \mathbb{C}E_{6,2} \oplus \mathbb{C}DE_4 \oplus \mathbb{C}DE_{4,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_4 = E_6 + 3DE_4,$$

$$E_2E_{4,2} = \frac{1}{21}E_6 + \frac{20}{21}E_{6,2} + 3DE_{4,2}$$

and

$$E_4E_{2,2} = \frac{5}{21}E_6 + \frac{16}{21}E_{6,2} + \frac{3}{2}DE_4.$$

The last three identities of theorem 6 are a direct consequence of the following ones:

$$E_2E_6 \in \widetilde{M}_8^{\leq 1}[\Gamma_0(1)] = M_8[\Gamma_0(1)] \oplus DM_6[\Gamma_0(1)] = \mathbb{C}E_8 \oplus \mathbb{C}DE_6,$$

and

$$\begin{aligned} E_{2,2}E_6,\ E_2E_{6,2} \in \widetilde{M}_8^{\leq 1}[\Gamma_0(2)] &= M_8[\Gamma_0(2)] \oplus DM_6[\Gamma_0(2)] \\ &= \mathbb{C}E_8 \oplus \mathbb{C}E_{8,2} \oplus \mathbb{C}\Delta_{8,2} \oplus \mathbb{C}DE_6 \oplus \mathbb{C}DE_{6,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_6 = E_8 + 2DE_6,$$

$$E_{2,2}E_6 = \frac{21}{85}E_8 + \frac{64}{85}E_{8,2} - \frac{2016}{17}\Delta_{8,2} + DE_6$$

and

$$E_2E_{6,2} = \frac{1}{85}E_8 + \frac{84}{85}E_{8,2} - \frac{504}{17}\Delta_{8,2} + 2DE_{6,2}.$$

5. O L

5.1. **Method.** For $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{N}^r$, $\mathbf{b} := (b_1, \dots, b_r) \in (2\mathbb{N}+1)^r$ and $\mathbf{N} := (N_1, \dots, N_r) \in \mathbb{N}^{*r}$ the sum $S[\mathbf{a}, \mathbf{b}, \mathbf{N}]$ defined in (1) is relied to the quasimodular forms *via* the function

$$D^{a_1}E_{b_1+1,N_1} \cdots D^{a_r}E_{b_r+1,N_r} \in \widetilde{M}_{b_1+\dots+b_r+r+2(a_1+\dots+a_r)}^{\leq a_1+\dots+a_r+t(\mathbf{b})}\left[\Gamma_0\left(\mathrm{lcm}(N_1, \dots, N_r)\right)\right] \tag{17}$$

where

$$t(\mathbf{b}) = \#\{i \in \{1, \dots, r\} : b_i = 1\}.$$

Since we always can consider that the coordinates of **a** are given in increasing order, let ℓ be the nonnegative integer such that $a_1 = \cdots = a_\ell = 0$ and $a_{\ell+1} \neq 0$ (we take $\ell = 0$ if **a** has all its coordinates positive). We consider the function

$$\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}} \coloneqq \prod_{j=1}^{\ell}(E_{b_j+1,N_1} - 1) \prod_{j=\ell+1}^{r} D^{a_j} E_{b_j+1,N_j}$$
$$\in \bigoplus_{k=b_{j+1}+\dots+b_r+r+2(a_{j+1}+\ldots+a_r)-j}^{b_1+\dots+b_r+r+2(a_{j+1}+\ldots+a_r)} \widetilde{M}_k^{\leq a_1+\ldots+a_r+t(\mathbf{b})}\left[\Gamma_0\left(\operatorname{lcm}(N_1,\ldots,N_r)\right)\right]$$

We have

$$\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}}(z) = (-2)^r \left[\prod_{j=1}^{r} \frac{b_j+1}{B_{b_j+1}}\right] \sum_{n=1}^{+\infty} S[\mathbf{a},\mathbf{b},\mathbf{N}] e^{2\pi i n z}.$$

The evaluation of $S[(0,1,1),(1,1,1)]$ is a consequence (by lemma 11) of

$$(E_1-1)(DE_2)^2 \in \mathbb{C}E_8 \oplus \mathbb{C}DE_6 \oplus \mathbb{C}D^2E_4 \oplus \mathbb{C}D^3E_2 \oplus \mathbb{C}E_{10} \oplus \mathbb{C}DE_8 \oplus \mathbb{C}D^2E_6 \oplus \mathbb{C}D^3E_4 \oplus \mathbb{C}D^4E_2.$$

The comparison of the first Fourier coefficients leads to

$$(E_2-1)(DE_2)^2 = -\frac{1}{5}D^2E_4 - 2D^3E_2 + \frac{2}{21}D^2E_6 + \frac{4}{5}D^3E_4 + 6D^4E_2.$$

Hence the evaluation of $S[(0,1,1),(1,1,1)]$ given in theorem 7.

The evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ is a consequence (by lemma 11) of

$$(E_2-1)^3(DE_2)^2 \in \mathbb{C}\Delta \oplus \mathbb{C}D\Delta \bigoplus_{i=1}^{7} \bigoplus_{j=0}^{7-i} \mathbb{C}D^jE_{2i}.$$

The comparison of the first Fourier coefficients leads to

$$\begin{aligned}(E_2-1)^3(DE_2)^2 = &-\frac{8}{35}\Delta + \frac{8}{35}D\Delta - 2D^3E_2 + 18D^4E_2 - \frac{216}{5}D^5E_2 + \frac{144}{5}D^6E_2\\ &- \frac{1}{5}D^2E_4 + \frac{12}{5}D^3E_4 - \frac{234}{35}D^4E_4 + \frac{171}{35}D^5E_4\\ &+ \frac{2}{7}D^2E_6 - \frac{9}{7}D^3E_6 + \frac{25}{21}D^4E_6 - \frac{1}{6}D^2E_8 + \frac{4}{15}D^3E_8 + \frac{2}{55}D^2E_{10}.\end{aligned}$$

Hence the evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ given in theorem 7.

We leave the proofs of propositions 8 and 9 to the reader. They are obtained from the linearisations of

$$(E_{2,2}-1)DE_{2,5} \in \widetilde{M}_4^{\leq 2}[\Gamma_0(5)] \oplus \widetilde{M}_6^{\leq 3}[\Gamma_0(10)] \tag{18}$$

and

$$DE_2DE_{2,5} \in \widetilde{M}_8^{\leq 5}[\Gamma_0(5)].$$

5.2. **Primitive forms of weight 6 and level 5 or 10.** For the evaluation of (18) we remark that $\Delta_{4,5}\Phi_{1,5}$ is a parabolic modular form of weight 6 and level 5. Since the dimension of these forms is 1, we have

$$\Delta_{6,5}(z) = \frac{1}{4}\left[\Delta(z)\Delta(5z)\right]^{1/6}\left[5E_2(5Z) - E_2(Z)\right]. \tag{19}$$

Equation (19) provides a way to compute the few needed values of $\tau_{6,5}$.

We also give expressions for $\Delta_{6,10,i}$ where $i \in \{1,2,3\}$. We shall use the second Hecke operator of level 10 given by

$$T_2 : \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imz} \mapsto \sum_{m\in\mathbb{Z}} \left[\sum_{\substack{d\in\mathbb{N}\\ d|(m,2)\\ (d,10)=1}} d^{k-1}\widehat{f}\left(\frac{2m}{d^2}\right)\right] e^{2\pi imz} = \sum_{m\in\mathbb{Z}} \widehat{f}(2m)e^{2\pi imz}.$$

The space of parabolic modular forms of weight 6 and level 10 has dimension 5. A basis is given by

$$\begin{aligned}
\Delta_{6,5}(z) &= [\Delta(z)\Delta(5z)]^{1/6}\,\Phi_{1,5}(z)\\
F_{6,5,2}(z) &= \Delta_{6,5}(2z)\\
F(z) &= 3\,[\Delta(z)\Delta(5z)]^{1/6}\,\Phi_{1,10}(z)\\
F_1(z) &= [\Delta(z)\Delta(5z)]^{1/6}\,\Phi_{1,2}(z)\\
F_2(z) &= T_2F(z).
\end{aligned}$$

To simplify the computations, we use an echelonised basis:

$$\begin{aligned}
V_1 &= -\frac{4}{15}\Delta_{6,5} + \frac{31}{10}F_{6,5,2} + \frac{15}{32}F + \frac{1}{96}F_1 + \frac{3}{80}F_2\\
&= e(z) + O(e(6z))\\
V_2 &= \frac{1}{20}\Delta_{6,5} + \frac{6}{5}F_{6,5,2} + \frac{1}{80}F_2\\
&= e(2z) + O(e(6z))\\
V_3 &= -\frac{1}{30}\Delta_{6,5} + \frac{7}{10}F_{6,5,2} + \frac{1}{32}F - \frac{1}{96}F_1 + \frac{1}{80}F_2\\
&= e(3z) + O(e(6z))\\
V_4 &= -\frac{1}{40}\Delta_{6,5} - \frac{1}{10}F_{6,5,2} - \frac{1}{160}F_2\\
&= e(4z) + O(e(6z))\\
V_5 &= \frac{1}{75}\Delta_{6,5} - \frac{11}{50}F_{6,5,2} - \frac{11}{800}F - \frac{1}{480}F_1 - \frac{3}{400}F_2\\
&= e(5z) + O(e(6z)).
\end{aligned}$$

We deduce

$$\Delta_{6,10,i} = V_1 + b_iV_2 + c_iV_3 + d_iV_4 + e_iV_5$$

since $\tau_{6,10,i}(1) = 1$. Now, from $\tau_{6,10,i}(4) = \tau_{6,10,i}(2)^2$, we get $d_i = b_i^2$ so that

$$\Delta_{6,10,i} = V_1 + b_iV_2 + c_iV_3 + b_i^2V_4 + e_iV_5.$$

Next, from $\tau_{6,10,i}(6) = \tau_{6,10,i}(2)\tau_{6,10,i}(3)$, and $\tau_{6,10,i}(8) = \tau_{6,10,i}(2)\tau_{6,10,i}(4)$, we respectively get

$$2c_i + b_ic_i + 2e_i = 10 - 2b_i - b_i^2 \tag{20}$$

and

$$8c_i - 8e_i = 24 + 16b_i + 6b_i^2 + b_i^3. \tag{21}$$

Equations (20) and (21) give either $b_i = -4$ or $b_i \neq -4$ and

$$c_i = 4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2$$

and

$$e_i = 1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3.$$

We first deal with the case $b_i \neq -4$. We then get

$$\Delta_{6,10,i} = V_1 + b_iV_2 + \left(4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2\right)V_3 + b_i^2V_4 + \left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right)V_5.$$

Using $\tau_{6,10,i}(10) = \tau_{6,10,i}(2)\tau_{6,10,i}(5)$, we obtain

$$b_i\left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right) = \\ -30 + 15b_i - 10\left(4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2\right) + 5b_i^2 + 6\left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right)$$

from what we get

$$b_i \in \left\{-4, 4, 1 - i\sqrt{31}, 1 + i\sqrt{31}\right\}.$$

The solution $b_i = -4$ is in this case not allowed whereas the solution $1 \pm i\sqrt{31}$ are not possible since the coefficients of a primitive form are totally real algebraic numbers. We thus obtain a first primitive form:

$$\Delta_{6,10,1} = V_1 + 4V_2 + 6V_3 + 16V_4 - 25V_5.$$

We assume now that $b_i = -4$ so that

$$\Delta_{6,10,i} = V_1 - 4V_2 + c_iV_3 + 16V_4 + (c_i + 1)V_5.$$

From $\tau_{6,10,i}(15) = \tau_{6,10,i}(3)\tau_{6,10,i}(5)$, we obtain $c_i = 24$ or $c_i = -26$. We hence get the two other primitive forms

$$\Delta_{6,10,2} = V_1 - 4V_2 + 24V_3 + 16V_4 + 25V_5$$

and

$$\Delta_{6,10,3} = V_1 - 4V_2 - 26V_3 + 16V_4 - 25V_5.$$

We deduce the following expressions:

$$\Delta_{6,10,1} = -\Delta_{6,5} + 16F_{6,5,2} + F + \frac{1}{4}F_2 \tag{22}$$

$$\Delta_{6,10,2} = -\frac{4}{3}\Delta_{6,5} + 8F_{6,5,2} + \frac{7}{8}F - \frac{7}{24}F_1 \tag{23}$$

$$\Delta_{6,10,3} = -\frac{1}{3}\Delta_{6,5} - 16F_{6,5,2} + \frac{1}{3}F_1 - \frac{1}{4}F_2. \tag{24}$$

Equations (22) to (24) provide a way to compute the few needed values of $\tau_{6,10,i}$ for $i \in \{1, 2, 3\}$.

5.3. **Primitive forms of weight 8 and level 5.** The method is the same as in §5.2 so we will be more brief. The space of parabolic forms of weight 8 and level 5 has dimension 3 and a basis is

$$G_1(z) = [\Delta(z)\Delta(5z)]^{1/3}$$
$$G_2(z) = [\Delta(z)\Delta(5z)]^{1/6}\Phi_{1,5}(z)^2$$
$$G_3 = -\frac{1}{24}\left[E_4, \Phi_{1,2}\right]_1$$

where $[\ ,\ ]_1$ is the Rankin-Cohen bracket here defined by

$$[E_4, \Phi_{1,2}]_1 = \frac{1}{2\pi i}(4E_4\Phi'_{1,5} - 2E'_4\Phi_{1,5})$$

(see [Zag92, part 1, §E] or [MR05, partie I, §6] for more details). We echelonise this basis by defining:

$$\begin{aligned}
W_1 &= \frac{46}{25}G_1 + \frac{82}{25}G_2 - \frac{3}{25}G_3 &&= e(z) + O(e(4z))\\
W_2 &= \frac{47}{375}G_1 - \frac{76}{375}G_2 + \frac{4}{375}G_3 &&= e(2z) + O(e(4z))\\
W_3 &= -\frac{41}{375}G_1 - \frac{19}{750}G_2 + \frac{1}{750}G_3 &&= e(3z) + O(e(4z)).
\end{aligned}$$

The primitive forms are then

$$\Delta_{8,5,i} = W_1 + b_i W_2 + c_i W_3.$$

From $\tau_{8,5,i}(4) = \tau_{8,5,i}(2)^2 - 2^7$ and $\tau_{8,5,i}(6) = \tau_{8,5,i}(2)\tau_{8,5,i}(3)$ we get

$$c_i = 78 + 2b_i - \frac{1}{2}b_i^2$$

$$(b_i + 14)(b_i^2 - 20b_i + 24) = 0.$$

Finally, defining v as one of the roots of $X^2 - 20X + 24$, we get

$$\Delta_{8,5,1} = \frac{16}{3}G_1 + \frac{22}{3}G_2 - \frac{1}{3}G_3 \tag{25}$$

$$\Delta_{8,5,2} = (12 - v)G_1 + G_2 \tag{26}$$

$$\Delta_{8,5,3} = (v - 8)G_1 + G_2. \tag{27}$$

Equations (25) to (27) provide a way to compute the few needed values of $\tau_{8,5,i}$ for $i \in \{1, 2, 3\}$.

R

[AAWa] A. Alaca, S. Alaca, and K.S. Williams. The convolution sum $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$. *Canad. Math. Bull.* To appear.

[AAWb] A. Alaca, S. Alaca, and K.S. Williams. The convolution sums $\sum_{\ell+24m=n} \sigma(\ell)\sigma(m)$ and $\sum_{3\ell+8m=n} \sigma(\ell)\sigma(m)$. Preprint.

[AAWc] A. Alaca, S. Alaca, and K.S. Williams. Evaluation of the convolution sums $\sum_{\ell+12m=n} \sigma(\ell)\sigma(m)$ and $\sum_{3\ell+4m=n} \sigma(\ell)\sigma(m)$. *Advances in Theoretical and Applied Mathematics*. To appear.

[AAWd] A. Alaca, S. Alaca, and K.S. Williams. Evaluation of the convolution sums $\sum_{\ell+18m=n} \sigma(\ell)\sigma(m)$ and $\sum_{2\ell+9m=n} \sigma(\ell)\sigma(m)$. Preprint.

[AW] S. Alaca and K.S. Williams. Evaluation of the convolution sums $\sum_{\ell+6m=n} \sigma(\ell)\sigma(m)$ and $\sum_{2\ell+3m=n} \sigma(\ell)\sigma(m)$. Preprint.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[Bes62] M. Besge. Extrait d'une lettre de M. Besge à M. Liouville. *J. Math. Pures Appl*, 7:256, 1862.

[CW05] N. Cheng and K.S. Williams. Evaluation of some convolution sums involving the sum of divisors function. *Yokohama Math. J.*, 52:39–57, 2005.

[DI95] Fred Diamond and John Im. Modular forms and modular curves. In *Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994)*, volume 17 of *CMS Conf. Proc.*, pages 39–133. Amer. Math. Soc., Providence, RI, 1995.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[Gla85] J. W. L. Glaisher. On the square of the series in which the coefficients are the sums of the divisors of the exponents. *Mess. Math.*, XV:156–163, 1885.

[HOSW02] James G. Huard, Zhiming M. Ou, Blair K. Spearman, and Kenneth S. Williams. Elementary evaluation of certain convolution sums involving divisor functions. In *Number theory for the millennium, II (Urbana, IL, 2000)*, pages 229–274. A K Peters, Natick, MA, 2002.

[Koi84] Masao Koike. On McKay's conjecture. *Nagoya Math. J.*, 95:85–89, 1984.

[KZ95] Masanobu Kaneko and Don Zagier. A generalized Jacobi theta function and quasimodular forms. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 165–172. Birkhäuser Boston, Boston, MA, 1995.

[Lah46] D. B. Lahiri. On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. I. *Bull. Calcutta Math. Soc.*, 38:193–206, 1946.

[Lah47] D. B. Lahiri. On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. II. *Bull. Calcutta Math. Soc.*, 39:33–52, 1947.

[LR05] Samuel Lelièvre and Emmanuel Royer. Orbitwise countings in $\mathcal{H}(2)$ and quasimodular forms, 2005. arXiv math.GT/0509205.

[LW05] M. Lemire and K.S. Williams. Evaluation of two convolution sums involving the sum of divisor functions. *Bull. Aust. Math. Soc.*, 73:107–115, 2005.

[Mel98] Giuseppe Melfi. On some modular identities. In *Number theory (Eger, 1996)*, pages 371–382. de Gruyter, Berlin, 1998.

[MR05] François Martin and Emmanuel Royer. Formes modulaires et périodes. In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 1–117. Soc. Math. France, Paris, 2005.

[Ram16] K.S. Ramanujan. On certain arithmetical functions. *Trans. Cambridge Philos. Soc.*, 22:159–184, 1916.

[Ran56] R. A. Rankin. The construction of automorphic forms from the derivatives of a given form. *J. Indian Math. Soc. (N.S.)*, 20:103–116, 1956.

[Ser77] Jean-Pierre Serre. *Cours d'arithmétique*. Presses Universitaires de France, Paris, 1977. Deuxième édition revue et corrigée, Le Mathématicien, No. 2.

[Shi72] Goro Shimura. Class fields over real quadratic fields and Hecke operators. *Ann. of Math. (2)*, 95:130–190, 1972.

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[Ste04] William Stein. Algorithms For Computing With Modular Forms, 2004. Preprint. Notes of a graduate course given at Harvard University, (Fall 2004).

[Wil] K.S. Williams. The convolution sum $\sum_{m<n/8} \sigma(m)\sigma(n-8m)$. *Pacific J. Math.* To appear.

[Wil05] Kenneth S. Williams. The convolution sum $\sum_{m<n/9} \sigma(m)\sigma(n-9m)$. *Int. J. Number Theory*, 1(2):193–205, 2005.

[Zag92] Don Zagier. Introduction to modular forms. In *From number theory to physics (Les Houches, 1989)*, pages 238–291. Springer, Berlin, 1992.

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